Teacher Notes

Topic C

Energy in simple and damped, driven oscillations.

We know that in SHM the displacement from equilibrium is $x = x_0 \sin(\omega_0 t + \phi)$ where ϕ is the phase and ω_0 is the natural frequency of the motion.

If we assume for concreteness that we are dealing with a mass-spring system, then $\omega_0 = \sqrt{\frac{k}{m}}$ where k is the spring constant.

The velocity is given by $v = \omega_0 x_0 \cos(\omega_0 t + \phi)$ and so $v_{max} = \omega_0 x_0$. Then, the maximum kinetic energy is

$$K_{max} = \frac{1}{2} m \omega_0^2 x_0^2$$

Similarly, the maximum potential energy is (recall that $\omega_0 = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_0^2$)

$$U_{max} = \frac{1}{2} k x_{max}^2 = \frac{1}{2} m \omega_0^2 x_0^2$$

This gives the result

$$\frac{K_{max}}{U_{max}} = \frac{\frac{1}{2} m \omega_0^2 x_0^2}{\frac{1}{2} m \omega_0^2 x_0^2} = 1$$

However, things are different for damped, driven oscillations.

We recall that if the angular frequency of the driving force is ω, then the displacement is $x = x_0 \sin(\omega t + \phi)$. Working as above we find that the kinetic energy involves the angular frequency of the driving force, $K_{max} = \frac{1}{2} m \omega^2 x_0^2$, whereas the potential energy is still $U_{max} = \frac{1}{2} k x_{max}^2 = \frac{1}{2} m \omega_0^2 x_0^2$, involving the natural angular frequency.

Thus, for damped driven oscillations:

$$\frac{K_{max}}{U_{max}} = \frac{\frac{1}{2} m \omega^2 x_0^2}{\frac{1}{2} m \omega_0^2 x_0^2} = \left(\frac{\omega}{\omega_0}\right)^2$$

This can be less than 1, equal to 1 or greater than 1 depending on the driving frequency.

It is also interesting to observe what happens to total energy.

In SHM we have

$K=\frac{1}{2}m\omega_0^2x_0^2\cos^2(\omega t+\phi)$ and $U=\frac{1}{2}m\omega_0^2x_0^2\sin^2(\omega t+\phi)$

This gives the familiar result:

$$E=K+U=\frac{1}{2}m\omega_0^2x_0^2\cos^2(\omega t+\phi)+\frac{1}{2}m\omega_0^2x_0^2\sin^2(\omega t+\phi)=\frac{1}{2}m\omega_0^2x_0^2$$

The total energy is constant.

However, in damped driven oscillations we have

$K=\frac{1}{2}m\omega^2x_0^2\cos^2(\omega t+\phi)$ and $U=\frac{1}{2}m\omega_0^2x_0^2\sin^2(\omega t+\phi)$

And so, the total energy is

$$E=K+U=\frac{1}{2}m\omega^2x_0^2\cos^2(\omega t+\phi)+\frac{1}{2}m\omega_0^2x_0^2\sin^2(\omega t+\phi)$$

which is not constant. The driving force is supplying energy as well as removing energy from the system. (See the Teacher Note "Resonance and energy conservation".) On the average though (a time average over a period)

$$<E>=<K>+<U>=\frac{1}{2}m\omega^2x_0^2<\cos^2(\omega t+\phi)>+\frac{1}{2}m\omega_0^2x_0^2<\sin^2(\omega t+\phi)>$$

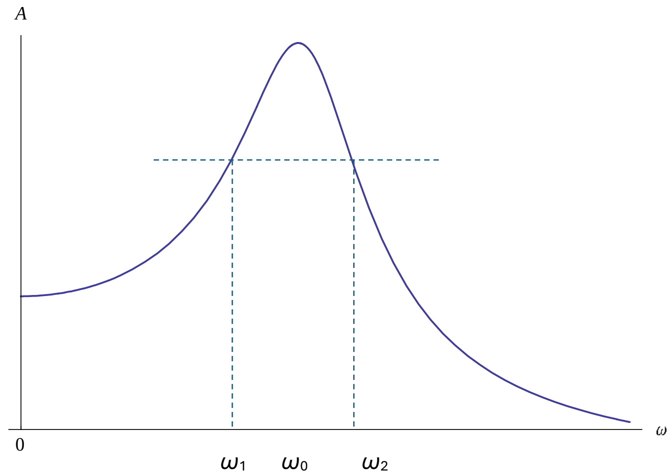
$$<E>=\frac{1}{2}m\omega^2x_0^2\times\frac{1}{2}+\frac{1}{2}m\omega_0^2x_0^2\times\frac{1}{2}=\frac{1}{4}m\left(\omega^2+\omega_0^2\right)x_0^2$$

which is a constant.

Examples

A damped system is driven at 2 different frequencies such that the amplitude of oscillations is the same. In which case is the time average of the total energy the greatest.

From the amplitude-frequency graph we know that one of the 2 frequencies must be below the natural frequency and the other above:



Hence the driving force with higher frequency has the greater average total energy.